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Hadronic Vacuum Polarization Contribution to $g - 2$ of the Leptons and $\alpha(M_Z)$

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Hadronic Vacuum Polarization Contribution to $g - 2$ of the Leptons and $\alpha(M_Z)$

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We review and compare recent calculations of hadronic vacuum polarization effects. In particular, we consider the anomalous magnetic moments $g - 2$ of the leptons and $\alpha(M_Z)$, the effective fine structure constant at the Z -resonance.

1. VACUUM POLARIZATION AND CHARGE SCREENING

Typically, charged particles in a collision of impact energy E interact electromagnetically with an effective charge which is the charge contained in the sphere of radius $r \simeq 1/E$ around the particles. As illustrated in Fig. 1 for one of the particles, the effective charge, due to vacuum polarization by virtual pair-creation, is larger than the classical charge which is seen in a large sphere ($r \rightarrow \infty$). This *charge screening* is a particular kind of charge renormalization.

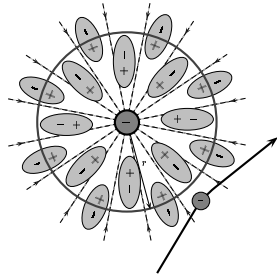


Figure 1. Vacuum polarization by virtual pair creation.

Commonly, Fig. 1 is represented by a Feynman diagram like the one in Fig. 2 contributing to muon scattering. Not surprising, the effective fine structure “constant” $\alpha(E)$ appears in many places in physics whenever the typical energy of a process is not in the classical regime. The major contribution to charge screening comes from light charged particle-antiparticle pairs of mass

$m \lesssim E/2$. While the lepton contributions can be easily calculated in QED perturbation theory the contribution of the strongly interacting quarks is not so easy to obtain. This is the issue of our discussion.

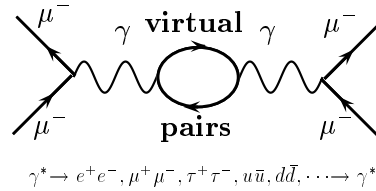


Figure 2. Feynman diagram describing the vacuum polarization in muon scattering.

1.1. Formal definition:

The effective QED coupling constant at scale \sqrt{s} may be written as

$$\alpha(s) = \frac{\alpha}{1 - \Delta\alpha(s)} \quad (1)$$

with

$$\Delta\alpha(s) = -4\pi\alpha\text{Re} [\Pi'_\gamma(s) - \Pi'_\gamma(0)] \quad (2)$$

where $\Pi'_\gamma(s)$ is the photon vacuum polarization function

$$\begin{aligned} & i \int d^4x e^{iq \cdot x} \langle 0 | T j_{em}^\mu(x) j_{em}^\nu(0) | 0 \rangle \\ &= -(q^2 g^{\mu\nu} - q^\mu q^\nu) \Pi'_\gamma(q^2) \end{aligned} \quad (3)$$

and $j_{em}^\mu(x)$ is the electromagnetic current.

*Report on work in collaboration with S. Eidelman [1].

1.2. Contributions:

• Leptons

Their contribution is calculable in perturbation theory. The free lepton loops are affected by small electromagnetic corrections only. In leading order one obtains:

$$\begin{aligned}\Delta\alpha_{\text{leptons}}(s) &= \sum_{\ell=e,\mu,\tau} \frac{\alpha}{3\pi} \left[-\frac{8}{3} + \beta_\ell^2 - \frac{1}{2}\beta_\ell(3 - \beta_\ell^2) \ln\left(\frac{1-\beta_\ell}{1+\beta_\ell}\right) \right] \\ &= \sum_{\ell=e,\mu,\tau} \frac{\alpha}{3\pi} \left[\ln\left(s/m_\ell^2\right) - \frac{5}{3} + O(m_\ell^2/s) \right] \quad (4) \\ &= 0.03142 \text{ for } s = M_Z^2\end{aligned}$$

where $\beta_\ell = \sqrt{1 - 4m_\ell^2/s}$.

• Quarks

The contribution of the five light quarks (u, d, s, c, b) is not calculable in perturbation theory. The free quark loops are strongly modified by strong interactions at low energy. The way out is the following: unitarity and the analyticity of $\Pi'_\gamma(s)$ allow us to write (Cabbibo, Gatto 1961[2])

$$\Delta\alpha_{\text{hadrons}}^{(5)}(s) = -\frac{\alpha s}{3\pi} \mathcal{P} \int_{4m_\pi^2}^{\infty} ds' \frac{R(s')}{s'(s' - s)} \quad (5)$$

where

$$R(s) \equiv \frac{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})}{\frac{4\pi\alpha^2(s)}{3s}} = 12\pi \text{Im}\Pi'_\gamma(s)$$

has been measured in e^+e^- -annihilation experiments up to energies above which we may calculate it in perturbative QCD. In other words

- $R(s)$ is known for low and moderately high s from the measurement of the total hadronic cross-section $e^+e^- \rightarrow \text{hadrons}$. This is of particular importance in the low energy region and for the resonance regions where non-perturbative physics comes into play.

- $R(s)$ may be reliably calculated for large s in *perturbative QCD* by virtue of the asymptotic freedom of QCD. The leading contribution is given by the sum over the squares of the charges Q_q of all quarks q

$$R(s) \simeq N_c \sum_q Q_q^2 (1 + O(\alpha_s/\pi))$$

with $N_c = 3$ the color factor.

We thus may evaluate the hadronic part of the vacuum polarization by utilizing e^+e^- -data (non-perturbative) for $\sqrt{s} \lesssim E_{\text{cut}} \sim 40\text{GeV}$ plus the perturbative tail. Of course the data exhibit experimental uncertainties which will allow us to estimate this contribution with limited accuracy only.

Before we continue to discuss the evaluation of the dispersion integral (5), let us remind the reader that the dispersion integral representation derives from basic properties valid for any quantum field theory:

□ *Unitarity* implies the *optical theorem*: The imaginary part of the forward scattering amplitude of an elastic process $A + B \rightarrow A + B$ is proportional to the sum over all possible final states $A + B \rightarrow \text{"anything"}$ (see Fig. 3)

$$\frac{\text{Im } T_{\text{forward}}(A + B \rightarrow A + B)}{\sqrt{\lambda(s, m_1^2, m_2^2)} \sigma_{\text{tot}}(A + B \rightarrow \text{anything})} =$$

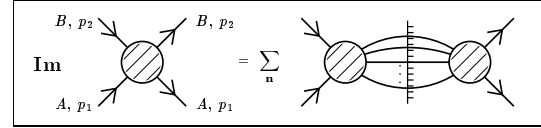


Figure 3. Optical theorem for scattering.

The corresponding relation for the photon propagator reads (see Fig. 4)

$$\text{Im}\Pi'_\gamma(s) = \frac{1}{12\pi} R(s)$$

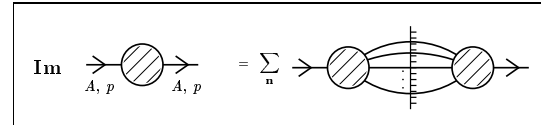


Figure 4. Optical theorem for propagation.

□ *Causality* implies *analyticity* which may be expressed in form of a so-called (subtracted) dispersion relation

$$\Pi'_\gamma(q^2) - \Pi'_\gamma(0) = \frac{q^2}{\pi} \int_0^\infty ds \frac{\text{Im}\Pi'_\gamma(s)}{s(s - q^2 - i\varepsilon)} .$$

The latter, together with the optical theorem, directly implies the validity of (5). Note that its validity is based on general principles and holds beyond perturbation theory. It is the basis of all non-perturbative evaluations of hadronic vacuum polarization effects in terms of experimental data.

1.3. Description of the e^+e^- -data.

In a recent reanalysis [1] of the hadronic vacuum polarization we have collected all published e^+e^- -data plus some unpublished data (see also [3]). The data sets obtained for different energy regions have been displayed in a number of figures in Ref. [1], and we will not repeat them here. The following **integration procedure** has been used for the evaluation of the dispersion integral:

- ❶ Take data as they are and apply trapezoidal rule (connecting data points by straight lines) for integration.
- ❷ To combine results from different experiments: i) integrate data for individual experiments and combine the results, ii) combine data from different experiments before integration and integrate combined “integrand”. Check consistency of the two possible procedures to estimate the reliability of the results.
- ❸ Error analysis: 1) statistical errors are added in quadrature, 2) systematic errors are added linearly for different experiments, 3) combined results are obtained by taking weighted averages. 4) all errors are added in quadrature for “independent” data sets. We assume this to be allowed in particular for different energy regions and/or different accelerators.
- ❹ Resonances have been parametrized by Breit-Wigner shapes with parameters taken from the Particle Data Tables [4].

Our recent update (Eidelman, Jegerlehner 1995)

$$\Delta\alpha_{\text{hadrons}}^{(5)}(M_Z^2) = 0.0280 \pm 0.0007$$

together with the leptonic term (4) yields an

Table 1

Comparison of estimates of $\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$

$\Delta\alpha_{\text{had}}^{(5)}(M_Z^2)$	Author (Year) [Ref.]
0.0285 (7)	Jegerlehner (86) [5]
0.0283(12)	Lynn et al. (87) [6]
0.0287 (9)	Burkhardt et al. (89) [7]
0.0282 (9)	Jegerlehner (91) [8]
0.0273 (4)	Martin, Zeppenfeld (94) [9]
0.0280 (7)	Eidelman, Jegerlehner (95) [1]
0.0280 (7)	Burkhardt, Pietrzyk (95) [10]
0.0275 (5)	Swartz (95) [11]
0.0289 (4)	Adel, Ynduráin (95) [12]

effective fine structure constant at the Z -resonance:

$$\alpha^{-1}(M_Z) = 128.89 \pm 0.09 .$$

1.4. Results for $\alpha(M_Z)$:

In Table 1 and Fig. 5 we show a comparison of results obtained by different authors. All values have been rescaled to $M_Z = 91.1887$ GeV. The small top quark and the W contributions are not included in $\alpha(M_Z)$. For a comparison of the earlier results [13] we refer to [5]. The differences are mainly due to a different treatment of the systematic errors and/or different model assumptions. Detail of our analysis are given in Table 2. Note that a reduction of the error would require a precision measurement of the $\sigma_{\text{tot}}(e^+e^- \rightarrow \text{hadrons})$ from 1 GeV to about 10 GeV.

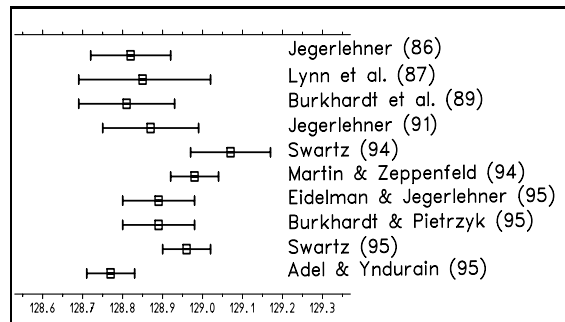


Figure 5. Different estimates of $\alpha(M_Z)^{-1}$.

A few remarks concerning the results presented in Tab. 1 are in order here. Updates seemed to be justified from time to time when new experimental data became available. While there have been

Table 2

Contributions to $\Delta\alpha_{\text{had}}^{(5)} \times 10^4$ and relative(rel) and absolute(abs) errors in percent.

final state	range (GeV)	contribution	contribution	rel %	abs %
ρ	(0.28, 0.81)	26.08 (0.29) (0.62)	Resonances:	46.61 (1.08)	2.3
ω	(0.42, 0.81)	2.93 (0.04) (0.08)			
ϕ	(1.00, 1.04)	5.08 (0.07) (0.12)			
J/ψ		11.34 (0.55) (0.61)			
Υ		1.18 (0.05) (0.06)			
hadrons	(0.81, 1.40)	13.83 (0.15) (0.79)	Background:	41.45 (4.11)	9.9
hadrons	(1.40, 3.10)	27.62 (0.32) (4.01)			
hadrons	(3.10, 3.60)	5.82 (0.30) (1.12)			
hadrons	(3.60, 9.46)	50.60 (0.24) (3.33)			
hadrons	(9.46, 40.0)	93.07 (0.86) (3.39)			
QCD	(40.0, ∞)	42.82 (0.00) (0.10)	40 GeV < E QCD	42.82 (0.10)	0.2
total		280.37(1.18) (6.43)	total	280.4 (6.54)	2.3

very little truly new experimental results, some groups have published updated results which were available before as preliminary data only. Examples are the ND results [14] up to 1.4 GeV, the DM2 results [15] between 1.35 and 2.3 GeV and the Crystal Ball results [16] between 5.0 and 7.4 GeV. New results from VEPP-4 [17], from 7.2 to 9.5 GeV, have been included as well. In addition in [1] we have made an effort to collect as much as possible all the available data.

Additional motivations for performing the update [1] were the following: the recent issues of the Review of Particle Properties [4] had some improvements on the resonance parameters and there was progress in calculating $R(s)$ at high energies in perturbative QCD. For the muon anomaly the study of the low energy end by means of chiral perturbation theory allowed us to reduce potential model-dependences of earlier approaches.

The update [7] attempted to estimate an “official” number which could be used by LEP experiments in their analyses. As compared to [5], a more conservative guess for the uncertainty in the continuum above the ρ up to the J/Ψ was made, and therefore, the uncertainty increased from 0.0007 to 0.0009. Later, the Crystal Ball (CB) collaboration carefully reanalyzed their old e^+e^- annihilation data and obtained $R(s)$ values substantially lower than the Mark I data and in agreement with other experiments (LENA).

The results were in much better agreement with perturbative QCD. The change of the data was mainly due to an up-to-date treatment of the QED radiative corrections and τ -subtraction. In Ref. [8] I included the updated results from CB and discarded the Mark I data, which systematically lie 28% higher, on average. Note that, because of the much larger systematic uncertainties of the Mark I data, the results are affected in a minor way if we exclude (50.60(0.24)(3.33)) or include (50.79(0.20)(3.20)) the Mark I data. The inclusion of the CB data lead to a reduction of previous results by -0.0005. While in Refs. [5,7,8] the ρ region was parametrized by an improved Gounaris-Sakurai formula, in our last update [1] we apply trapezoidal rule integration and include the final data from VEPP-2M OLYA, ND and CMD up to 1.4 GeV. This extended analysis confirmed our previous results. From 1.35 to 2.3 GeV new data from ORSAY DCI DM2 lead to a reduction of the error in this region. The error now agrees with the earlier estimate in [5].

While the analysis [10] is similar in spirit to ours and reproduced our result, the estimate [9] is biased by believing in perturbative QCD down to 3 GeV; some experimental data used in the J/Ψ and Υ resonance regions are rescaled using perturbative QCD. In contrast to this estimate which yields a low value, the estimate [12], which also relies on perturbative QCD, finds a large value; both estimates result in much lower errors than

what can be justified by the data alone. In the analysis of [11] data are fitted before integration. This requires a guess of the functional form of the integrand which is then fitted to the data. In addition one has to assume some kind of correlation matrix between the data points. One consequence of the method applied is that the inclusion of one additional unpublished data point [18] led to a shift of the result by 1σ .

In Fig. 6 we show the result for $\alpha(-Q^2)$ as a function of $E = |Q|$ for low $t = -Q^2$ in the space-like region, which is relevant for the t -channel contribution to Bhabha scattering. Note the

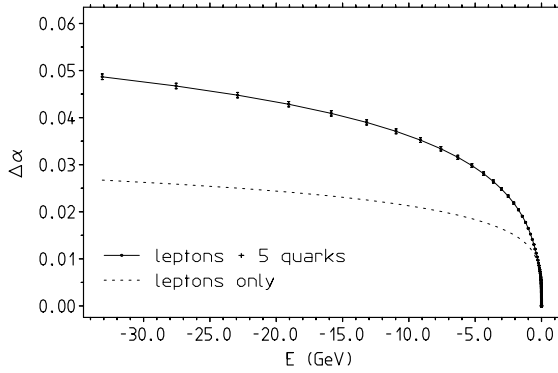


Figure 6. $\Delta\alpha(-Q^2)$ in the spacelike region.

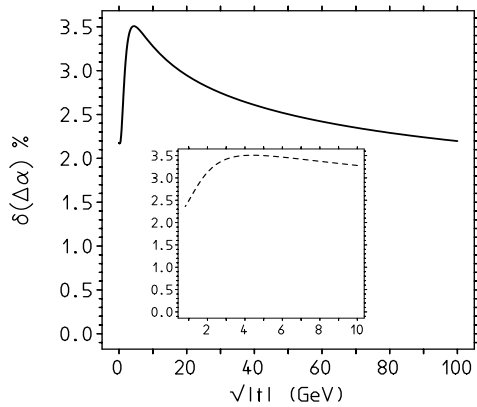


Figure 7. Uncertainty of $\Delta\alpha(t)$.

dramatic increase of the effective charge at low spacelike momenta. This shows that the classical limit is difficult to reach in a scattering experiment. A clean measurement of the running

of $\alpha(-Q^2)$ is possible at LEP by an appropriate analysis of the small angle Bhabha scattering data. The values of Q^2 should be low enough such that the t -channel contribution is clearly dominant. In Fig. 7 we show the uncertainty in % of our evaluation of the running charge in the low energy region.

1.5. Parametrizations.

We have performed fits of $\Delta\alpha_{\text{had}}^{(5)}(s)$. The parametrizations² and the best fit parameters we found are the following:

For $-(2\text{GeV})^2 < s < (0.25\text{GeV})^2$ the best fit is

$$\Delta\alpha_{\text{had}}^{(5)}(s) = c_1 \left\{ \ln|1 - c_2 s| + c_2 \frac{c_3 s}{c_3 - s} \right\} - l_1 \frac{c_3 s}{c_3 - s} + c_4 (s/s_0)^2$$

with $s_0 = -(2\text{GeV})^2$, $l_1 = 9.3055 \times 10^{-3} \text{ GeV}^{-2}$,
 $c_1 = 2.2694240 \times 10^{-3}$, $c_2 = 8.073429 \text{ GeV}^{-2}$,
 $c_3 = 0.1636393 \text{ GeV}^2$, $c_4 = -3.35455 \times 10^{-5}$.
 In the range $-(20\text{GeV})^2 < s < -(2\text{GeV})^2$ we find

$$\Delta\alpha_{\text{had}}^{(5)}(s) = c_1 \left\{ \ln|1 - c_2 s| + c_2 \frac{c_3 s}{c_3 - s} \right\} - l_1 \frac{c_3 s}{c_3 - s} + c_4 (s/s_0)$$

with $s_0 = -(20\text{GeV})^2$, $l_1 = 9.3055 \times 10^{-3} \text{ GeV}^{-2}$,
 $c_1 = 2.8668314 \times 10^{-3}$, $c_2 = 0.3514608 \text{ GeV}^{-2}$,
 $c_3 = 0.5496359 \text{ GeV}^2$, $c_4 = 1.989233 \times 10^{-4}$.
 At higher energies, in the ranges $E_{\text{min}} < E < E_{\text{max}}$, we have

$$\Delta\alpha_{\text{had}}^{(5)}(s) = c_1 + c_2 (\ln(s/s_0)) + c_3 (s_0/s - 1) + c_4 ((s_0/s)^2 - 1))$$

²Fortran routines `hadr5.f` and `alphaQED.f` for calculating $\Delta\alpha_{\text{had}}^{(5)}(s)$ and $\alpha(s)$, respectively, are available from WWW <http://www.ifh.de/~fjeger/>

where $s = E|E|$ and $s = E_0|E_0|$ with

E_{\min}	250	80	40
E_{\max}	1000	250	80
E_0	1000	91.1888	80
$c_1 \times 10^2$	4.2092260	2.8039809	2.7266588
$c_2 \times 10^3$	2.9233438	2.9373798	2.9285045
$c_3 \times 10^3$	-0.3296691	-2.8432352	-4.7720564
$c_4 \times 10^4$	0.0034324	-5.2537734	7.7295507
E_{\min}	-200	-1000	
E_{\max}	-20	-200	
E_0	-100	-1000	
$c_1 \times 10^2$	2.8526291	4.2069394	
$c_2 \times 10^3$	2.9520725	2.9253566	
$c_3 \times 10^3$	-2.7906310	-0.6778245	
$c_4 \times 10^4$	0.6417453	0.0932141	

At the higher energies, $E \geq 40$ GeV, we may fit the hadronic contribution using the following *effective parameters* in a $O(\alpha_s)$ perturbative QCD formula: effective quark masses: $m_{u,d,s,c,b} = 0.067, 0.089, 0.231, 1.299, 4.500$ GeV together with an effective $\alpha_s = 0.102$. Thus

$$\Delta\alpha_{\text{had}}^{(5)}(s) = -\frac{\alpha}{9\pi} (1 + \alpha_s^{\text{eff}}/\pi) (h(y_d) + h(y_s) + h(y_b) + 4(h(y_u) + h(y_c)))$$

where

$$h(y) = 5/3 + y - (1 + y/2) g(y)$$

$$g(y) = \begin{cases} 2\sqrt{y-1} \arctan(1/\sqrt{y-1}) & \text{for } y > 1 \\ \sqrt{1-y} \ln(|\frac{1+\sqrt{1-y}}{1-\sqrt{1-y}}|) & \text{for } y < 1 \end{cases}$$

with $y_i = 4m_i^2/s$.

2. VACUUM POLARIZATION CONTRIBUTION TO $g - 2$

The leading hadronic contribution to $g - 2$ is given by the diagram Fig. 8, where the blob represents the one-particle irreducible contribution to the photon propagator. To evaluate the diagram Fig. 8 we can make use of the representation (5) for the vacuum polarization function. Again one obtains a dispersion integral (Bouchiat, Michel 1961, Durand III 1962, Gourdin, de Rafael 1969)

$$\mathbf{a}_\mu \equiv \frac{\mathbf{g}_\mu - 2}{2} = \left(\frac{\alpha \mathbf{m}_\mu}{3\pi}\right)^2 \int_{4m_\pi^2}^{\infty} ds \frac{\mathbf{R}(s) \hat{\mathbf{K}}(s)}{s^2} \quad (6)$$

which may be evaluated in terms of the experimentally accessible quantity

$$\mathbf{R}(s) = \frac{\sigma_{\text{tot}}(e^+e^- \rightarrow \gamma^* \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^-)} \quad .$$

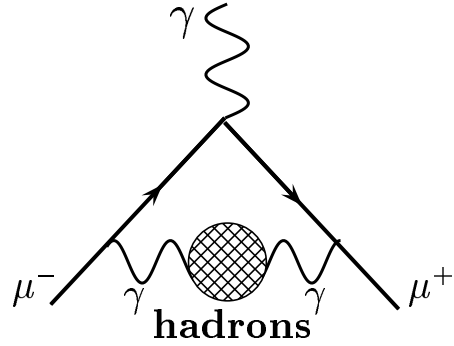


Figure 8. Leading vacuum polarization contribution to $g - 2$.

The kernel $\hat{K}(s)$ of (6), depicted in Fig. 9, is a smooth *bounded function*.

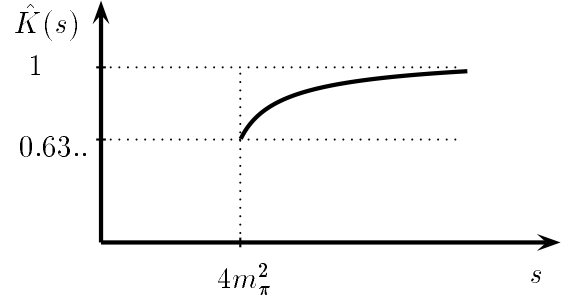


Figure 9. The kernel $\hat{K}(s)$ as a function of energy.

Note that the energy denominator $1/s^2 = 1/E^4$ *dramatically enhances the non-perturbative low energy contribution*

2.1. Results for a_μ^{had} :

The present status of the hadronic vacuum contribution to $g - 2$ is summarized in Table 4 and Fig. 10. Only the leading hadronic vacuum polarization diagram is included. Our estimate is

$$a_\mu^{\text{had}} = 702.35(15.28) \times 10^{-10} \quad .$$

Our values for a_e^{had} and a_τ^{had} are given in Table 5.

Table 3
 $a_\mu^{\text{had}} \cdot 10^{10}$ and relative(rel) and absolute(abs) errors in percent.

final state	range (GeV)	$a_\mu^{\text{had}} \cdot 10^{10}$	rel %	abs %
ρ	(0.28, 0.81)	426.66 (5.61) (10.62)	2.8%	1.7%
ω	(0.42, 0.81)	37.76 (0.45) (1.02)	3.0%	0.2%
ϕ	(1.00, 1.04)	38.55 (0.54) (0.89)	2.7%	0.1%
J/ψ		8.60 (0.41) (0.40)	6.7%	0.1%
Υ		0.10 (0.00) (0.01)	6.7%	0.0%
hadrons	(0.81, 1.40)	112.85 (1.33) (5.49)	5.0%	0.8%
hadrons	(1.40, 3.10)	56.43 (0.45) (7.22)	12.8%	1.0%
hadrons	(3.10, 3.60)	4.47 (0.23) (0.86)	19.9%	0.1%
hadrons	(3.60, 9.46)	14.06 (0.07) (0.90)	6.5%	0.1%
hadrons	(9.46, 40.0)	2.70 (0.03) (0.13)	4.9%	0.0%
QCD	(40.0, ∞)	0.16 (0.00) (0.00)	0.2%	0.0%
total		702.35 (5.85) (14.09)	2.2 %	2.2%

Table 4
 Various estimates of $a_\mu^{\text{had}} \times 10^{10}$

$a_\mu^{\text{had}} \times 10^{10}$	Author (Year) [Ref.]
650 (50)	Gourdin, de Rafael (69) [22]
680 (90)	Bramon, Etim, Greco (72) [23]
663 (85)	Barger, Long, Olsen (75) [24]
702 (80)	Calmet et al. (78) [25]
684 (11)	Barkov et al. (85) [26]
707 (6)(16)	Kinoshita et al. (85) [27]
710(10)(5)	Casas et al. (85) [28]
705 (6)(5)	Martinovič, Dubnička (90) [29]
724 (7)(26)	Jegerlehner (91) [30]
699 (4)(2)	Dubnicková et al. (92) [31]
702 (6)(14)	Eidelman, Jegerlehner (95) [1]
711 (10)	Adel, Ynduráin (95) [12]
702 (8)(13)	Worstell, Brown (95) [32]

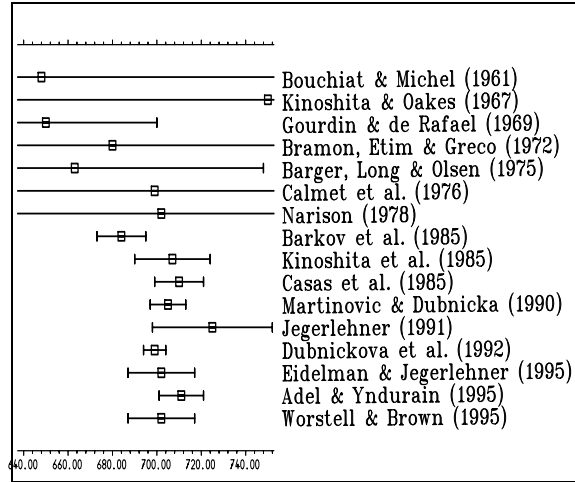


Figure 10. Evaluations of $a_\mu^{\text{hadrons}} \times 10^{10}$ by different authors.

3. CONCLUSION AND OUTLOOK

► Muon anomalous magnetic moment: The present uncertainty due to hadronic vacuum polarization is

$$(g-2)_\mu : \delta a_\mu \sim 16 \times 10^{-10}$$

This can be reduced to 8×10^{-10} by ongoing measurements at VEPP 2M in Novosibirsk [35] and in a forthcoming experiment at DAΦNE in Frascati [36]. This has to be compared with the ex-

perimental precision of the forthcoming BNL experiment E821 at Brookhaven [33]. The present experimental precision is 84×10^{-10} was obtained in the CERN experiment [34]. A look at Table 3 shows that a measurement of $e^+e^- \rightarrow \text{hadrons}$ for 1.5 GeV to 3.6 GeV to 1% accuracy is urgently needed in order to be able to fully exploit the BNL measurement of a_μ .

► Bhabha scattering: the hadronic uncertainty is not a limiting factor in small angle Bhabha scattering (luminosity monitoring). The esti-

Table 5

Contributions to $a_e^{\text{had}} \cdot 10^{14}$ and $a_\tau^{\text{had}} \cdot 10^8$.

final state	range (GeV)	$a_e^{\text{had}} \cdot 10^{14}$ (stat) (syst)	$a_\tau^{\text{had}} \cdot 10^8$ (stat) (syst)
ρ	(0.28, 0.81)	118.57 (1.59) (2.97)	137.42 (1.61) (3.33)
ω	(0.42, 0.81)	10.07 (0.12) (0.27)	14.62 (0.18) (0.39)
ϕ	(1.00, 1.04)	9.86 (0.14) (0.23)	20.36 (0.29) (0.47)
J/ψ		2.04 (0.10) (0.09)	12.80 (0.60) (0.61)
Υ		0.02 (0.00) (0.00)	0.24 (0.01) (0.01)
hadrons	(0.81, 1.40)	29.16 (0.35) (1.41)	56.22 (0.62) (3.00)
hadrons	(1.40, 3.10)	13.70 (0.11) (1.75)	56.32 (0.54) (7.60)
hadrons	(3.10, 3.60)	1.06 (0.06) (0.20)	6.66 (0.35) (1.28)
hadrons	(3.60, 9.46)	3.31 (0.02) (0.21)	26.43 (0.13) (1.70)
hadrons	(9.46, 40.0)	0.63 (0.01) (0.03)	6.78 (0.06) (0.33)
QCD	(40.0, ∞)	0.04 (0.00) (0.00)	0.45 (0.00) (0.00)
total		188.47 (1.65) (3.75)	338.30 (1.97) (9.12)

mated uncertainty is $\lesssim 0.06\%$.

► Precision measurements at LEP and SCL: the hadronic uncertainty is close to become the limiting factor for precision measurements of some observables like

$$\sin^2 \Theta_{\text{eff}}^\ell : \delta \sin^2 \Theta \sim 0.00025$$

This compares to the present accuracy of $\delta \sin^2 \Theta \sim 0.00030$ of LEP and SLC data.

► In principle one could try to do a direct measurement of $(g-2)_\mu^{\text{had}}$ and $\alpha(M_Z)$. Unfortunately, in this case one would lose the most important predictions and in addition the result would be model dependent !.

► Reliable theoretical calculations are not in sight.

► A real breakthrough in improving on the hadronic uncertainty will require dedicated efforts in high precision measurements of $R(s)$ in a wide energy range.

4. APPENDIX: Analytic expressions.

In many cases, in some interval $4m_\pi^2 \leq s_1 < s_2 \leq \infty$, one may approximate $R(s)$ by simple analytic expressions and the dispersion integral

$$\begin{aligned} \Delta\alpha_{\text{had}}(s) &= \frac{-s}{4\pi^2\alpha} \mathcal{P} \int_{s_1}^{s_2} ds' \frac{\sigma_{\text{had}}(s')}{s' - s} \\ &= \frac{\alpha}{3\pi} \mathcal{P} \int_{s_1}^{s_2} ds' \left\{ \frac{1}{s'} - \frac{1}{s' - s} \right\} R(s') , \end{aligned}$$

may be performed analytically.

We consider a few examples in the following.

• Linear extrapolation

For example, for non-resonant contributions (neighboring) data points may be approximated by a linear function

$$R(s) = a + b\sqrt{s}$$

in the c.m. energy which gives a contribution³

$$\begin{aligned} \Delta\alpha_{\text{had}}(s) &= \frac{\alpha}{3\pi} \left\{ 2a \ln \frac{W_2}{W_1} - (a + bW) \ln \frac{W_2 - W}{W_1 - W} \right. \\ &\quad \left. - (a - bW) \ln \frac{W_2 + W}{W_1 + W} \right\} \\ &= \frac{\alpha}{3\pi} \left\{ 2a \ln \frac{W_2}{W_1} - a \ln \left| \frac{s_2 - s}{s_1 - s} \right| - bf(W) \right\} , \end{aligned}$$

where $W_i = \sqrt{s_i}$, $W = \sqrt{s}$ and

$$f(W) = \begin{cases} W (\ln |\frac{W_2 - W}{W_1 - W}| - \ln |\frac{W_2 + W}{W_1 + W}|); & q^2 = W^2 > 0 \\ 2|W|(\arctan \frac{|W|}{W_2} - \arctan \frac{|W|}{W_1}); & q^2 = W^2 < 0. \end{cases}$$

• Narrow width resonance

The contribution from a zero width resonance

$$\sigma_{\text{NW}}(s) = \frac{12\pi^2}{M_R} \Gamma_{e^+e^-} \delta(s - M_R^2)$$

³Eq. (7) of Ref. [7] contains several misprints

or

$$R_{NW}(s) = \frac{9\pi M_R}{\alpha^2} \Gamma_{e^+e^-} \delta(s - M_R^2)$$

is given by

$$\Delta\alpha_{res}(s) = \frac{3\Gamma_{e^+e^-}}{\alpha M_R} \frac{s}{s - M_R^2}$$

which in the limit $|s| \gg M_R^2$ becomes

$$\Delta\alpha_{res}(s) = \frac{3\Gamma_{e^+e^-}}{\alpha M_R}.$$

• Breit-Wigner resonance

The contribution from a Breit-Wigner resonance

$$\sigma_{BW}(s) = \frac{3\pi}{s} \frac{\Gamma \Gamma_{e^+e^-}}{(\sqrt{s} - M_R)^2 + \frac{\Gamma^2}{4}}$$

or

$$R_{BW}(s) = \frac{9}{4\alpha^2} \frac{\Gamma \Gamma_{e^+e^-}}{(\sqrt{s} - M_R)^2 + \frac{\Gamma^2}{4}}$$

is given by

$$\Delta\alpha_{res}(s) = \frac{3\Gamma \Gamma_{e^+e^-}}{4\pi\alpha} \{I(0) - I(W)\}$$

where

$$I(W) = \frac{1}{2ic} \left\{ \frac{1}{W - M_R - ic} \left(\ln \frac{W_2 - W}{W_1 - W} - \ln \frac{W_2 - M_R - ic}{W_1 - M_R - ic} \right) - \frac{1}{W + M_R + ic} \left(\ln \frac{W_2 + W}{W_1 + W} - \ln \frac{W_2 - M_R - ic}{W_1 - M_R - ic} \right) - h.c. \right\}$$

with $c = \Gamma/2$. This may be written as

$$\Delta\alpha_{res}(s) = \frac{3\Gamma_{e^+e^-}}{\pi\alpha M_R} \frac{M_R^2}{M_R^2 + c^2} \frac{1}{(s - M_R^2 + c^2)^2 + M_R^2 \Gamma^2} \left\{ s(s - M_R^2 + 3c^2) \left(\pi - \arctan \frac{c}{W_2 - M_R} - \arctan \frac{c}{M_R - W_1} \right) - \frac{\Gamma}{4M_R} \left[s(s - 3M_R^2 + c^2) \ln \frac{(W_2 - M_R)^2 + c^2}{(W_1 - M_R)^2 + c^2} + (s + M_R^2 + c^2)(M_R^2 + c^2) \ln \left| \frac{s_2 - s}{s_1 - s} \right| + 2M_R(M_R^2 + c^2)f(W) + 2((s - M_R^2 + c^2)^2 + M_R^2 \Gamma^2) \ln \frac{W_2}{W_1} \right] \right\}$$

with $f(W)$ given above. For $W_1 \ll M_R \ll W_2$ and $\Gamma \ll M_R$ this may be approximated by

$$\Delta\alpha_{res}(s) = \frac{3\Gamma_{e^+e^-}}{\alpha M_R} \frac{s(s - M_R^2 + 3c^2)}{(s - M_R^2 + c^2)^2 + M_R^2 \Gamma^2}$$

which agrees with Eqs. (3) and (4) in the limits $\Gamma^2 \ll |s - M_R^2|, M_R^2$ and $|s| \gg M_R^2$, respectively.

• Breit-Wigner resonance: Field theory version

Finally, we consider a field theoretic form of a Breit-Wigner resonance obtained by the Dyson summation of a massive spin 1 transversal part of the propagator in the approximation that the imaginary part of the self-energy yields the width by $\text{Im}\Pi_V(M_V^2) = M_V \Gamma_V$ near resonance.

$$\sigma_{BW}(s) = \frac{12\pi}{M_R^2} \frac{\Gamma_{e^+e^-}}{\Gamma} \frac{s\Gamma^2}{(s - M_R^2)^2 + M_R^2 \Gamma^2}$$

which yields

$$\Delta\alpha_{res}(s) = \frac{3\Gamma_{e^+e^-}}{\pi\alpha M_R} \frac{s(s - M_R^2 - \Gamma^2)}{(s - M_R^2)^2 + M_R^2 \Gamma^2} \left\{ \left(\pi - \arctan \frac{\Gamma M_R}{s_2 - M_R^2} - \arctan \frac{\Gamma M_R}{M_R^2 - s_1} \right) - \frac{\Gamma}{M_R} \frac{s}{(s - M_R^2 - \Gamma^2)} \left(\ln \left| \frac{s_2 - s}{s_1 - s} \right| - \ln \left| \frac{s_2 - M_R^2 - iM_R \Gamma}{s_1 - M_R^2 - iM_R \Gamma} \right| \right) \right\}$$

and reduces to

$$\Delta\alpha_{res}(s) = \frac{3\Gamma_{e^+e^-}}{\alpha M_R} \frac{s(s - M_R^2 - \Gamma^2)}{(s - M_R^2)^2 + M_R^2 \Gamma^2}$$

for $s_1 \ll M_R^2 \ll s_2$ and $\Gamma \ll M_R$. Again we have the known limits for small Γ and for large $|s|$. For broad resonances the different parametrizations of the resonance in general yield very different results. Therefore, it is important to know how a resonance was parametrized to get the resonance parameters like M_R and Γ . For narrow resonances as considered in our application results are not affected in a relevant way by using different parametrizations.

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